Early Journal Content on JSTOR, Free to Anyone in the World

This article is one of nearly 500,000 scholarly works digitized and made freely available to everyone in the world by JSTOR.

Known as the Early Journal Content, this set of works include research articles, news, letters, and other writings published in more than 200 of the oldest leading academic journals. The works date from the mid-seventeenth to the early twentieth centuries.

We encourage people to read and share the Early Journal Content openly and to tell others that this resource exists. People may post this content online or redistribute in any way for non-commercial purposes.

Read more about Early Journal Content at http://about.jstor.org/participate-jstor/individuals/early-journal-content.

JSTOR is a digital library of academic journals, books, and primary source objects. JSTOR helps people discover, use, and build upon a wide range of content through a powerful research and teaching platform, and preserves this content for future generations. JSTOR is part of ITHAKA, a not-for-profit organization that also includes Ithaka S+R and Portico. For more information about JSTOR, please contact support@jstor.org.

If f denote the total order of intersection of the sheet or sheets of S with the two sheets of Σ all along the double line of the latter surface, f_1 denote the component of that total order arising in the first sheet and f_2 likewise the component of that total order found in the second sheet, f'_s ($s = 1$ or 2) denote that component of f_s due to simple or multiple intersection alone (with regard to the sheet indicated by s, or both sheets, if s is not written), regardless of contact of sheets, and f''_s likewise the component due to such contact between sheets of S and Σ, while g_1 and g_2 in like manner denote the total orders, in the residual intersection, of the generators brought in by ω_τ in the first and the second sheets of Σ respectively, then will

$$f = f_1 + f_2 = f'_1 + f'_2 + f''_1 + f''_2 = f' + f'' \text{ and } f'_1 = f'_2,$$

while, in general,

$$f = \tfrac{1}{2}(p + \tau) - q + 3, \quad f_1 = \tfrac{1}{2}(p + \tau) - q + 1, \quad f_2 = \theta_1 + 2, \quad g_1 = \tau - \theta_1,$$

and $g_2 = \theta_1$ (which last, g_2, is included in f_2 and will henceforth be omitted save as it appears there). The general case may, then, be written thus:

$$m' = \tfrac{1}{2}(p + \tau) + 1, \quad f_1 = \tfrac{1}{2}(p + \tau) - q + 1, \quad f_2 = \theta_1 + 2, \quad g_1 = \tau - \theta_1,$$

where $\omega = \lambda^2 . \omega_\tau$, $p + \tau \geqq 2q$ and *even*, and $\theta_1 \leqq \tfrac{1}{2}(p + \tau) - q - 1$. And here will

$$f'_1 = f'_2 = 1, \qquad f''_1 = 0, \qquad f''_2 = 1,$$

if $p + \tau = 2q$; while

$$f'_1 = f'_2 = \theta_1 + 2, \quad f''_1 = \tfrac{1}{2}(p + \tau) - q - \theta_1 - 1, \quad f''_2 = 0,$$

if $p + \tau > 2q$.

The cases where S is the surface of lowest order which can be passed through the given curve on Σ, can now be solved readily.

If $\phi \equiv F(\lambda^2, \lambda\mu, \mu\nu, \mu^2 - \lambda\nu)$, the case is that of complete intersection already considered, pages 185 and 186, and may be expressed thus:

I′. $\phi \equiv F(\lambda^2, \lambda\mu, \mu\nu, \mu^2 - \lambda\nu)$, $p = 2q$, $m' = \tfrac{1}{2}p$, $f = g_1 = 0$, $\omega \equiv 1$.

If $\phi \not\equiv F(\lambda^2, \lambda\mu, \mu\nu, \mu^2 - \lambda\nu)$ and $p \geqq 2q$ and *even*, the results above show that τ can be given the value zero, leading to the general case

I. $\phi \not\equiv F(\lambda^2, \lambda\mu, \mu\nu, \mu^2 - \lambda\nu)$, $p \geqq 2q$, p *even*, $m' = \tfrac{1}{2}p + 1$,
$f_1 = \tfrac{1}{2}p - q + 1$, $f_2 = 2$, $g_1 = 0$, $\omega \equiv \lambda^2$,

while

$$f_1' = f_2' = 1, \quad f_1'' = 0, \quad f_2'' = 1, \text{ if } p = 2q,$$

but $$f_1' = f_2' = 2, \quad f_1'' = \tfrac{1}{2}p - q - 1, \quad f_2'' = 0, \text{ if } p > 2q.$$

If $p \geqq 2q$ and *odd*, an important special case arises when

$$\phi \equiv \lambda \,.\, F_1(\lambda^2, \lambda\mu, \mu\nu, \mu^2 - \lambda\nu) + \mu \,.\, F_2(\lambda^2, \lambda\mu, \mu\nu, \mu^2 - \lambda\nu);$$

for multiplication by λ allows the substitution to be entirely performed at once, giving $S \equiv x \,.\, F_1(x, y, z, s) + y \,.\, F_2(x, y, z, s) = 0$, where $m' = \frac{1}{2}(p+1)$ and S contains the line x, y once, so that this line occurs twice in the residual intersection of the two surfaces. In order that ϕ may not be reducible by the factor λ, it is necessary that some term in F_2 contain ν to the degree $\frac{1}{2}(p-1)$; hence, this case can arise only when $q = \frac{1}{2}(p-1)$, and may be described as

II′. $\phi \equiv \lambda F_1 + \mu F_2, \quad p = 2q + 1, \quad m' = \frac{1}{2}(p+1),$
$f_1 = f_2 = 1, \quad g_1 = 0, \quad \omega \equiv \lambda$ and $f_1'' = f_2'' = 0.$

Geometrically, this case includes only those curves which have q pairs of branches along the double line of Σ; i. e., at each of the q points where the curve ϕ meets the double line in the second sheet, a branch of the curve meets that line at the same point in the first sheet; but, since $p - q = q + 1$, there is one more branch in the first sheet in the neighborhood of the double line than there are branches in the second sheet in the same neighborhood, and, consequently, the introduction of the double line in the second sheet completes the intersection by forming a pair of branches with the single or superfluous branch in the first sheet. This case forms the nearest analogy, when p is odd, to the case of complete intersection, where p is even.

When $\phi = 0$, with $p \geqq 2q$ and odd, does not have the special form just considered, it is evident that the surface S of lowest order is obtained by giving to τ the value unity, leading to the case

II. $\phi \not\equiv \lambda F_1 + \mu F_2, \quad p \geqq 2q, p$ *odd*, $\quad m' = \frac{1}{2}(p+3), \quad f_1 = \frac{1}{2}(p+3) - q,$
$f_2 = \theta_1 + 2, \quad g_1 = 1 - \theta_1, \quad \omega \equiv \lambda^2 . \omega_1,$

and $\quad f_1' = f_2' = 1, \quad f_1'' = 0, \quad f_2'' = 1, \text{ if } p + 1 = 2q,$

but $\quad f_1' = f_2' = \theta_1 + 2, \quad f_1'' = \frac{1}{2}(p-1) - q - \theta_1, \quad f_2'' = 0, \text{ if } p + 1 > 2q.$

If $p < 2q$, τ must have a value greater than zero in the general case, since the condition $p + \tau \geqq 2q$ must be satisfied. The surface S will be of the lowest

possible order when τ is given its lowest possible value $\tau = 2q - p$, leading to the case

$$\text{III.}\quad p \leqq 2q,\quad m' = q + 1,\quad f_1 = 1,\quad f_2 = 2,\quad g_1 = 2q - p,\quad \omega = \lambda^2.\omega_\tau,$$

and $\theta_1 = 0$, since the condition $\theta_1 \leqq \frac{1}{2}(p + \tau) - q - 1$ here becomes impossible of satisfaction by any positive integer. But special cases, in which a surface S of order lower than $q + 1$ can be passed through the curve ϕ, may occur here. If $\phi \equiv \lambda \,.\, F_1(\lambda^2, \lambda\mu, \mu\nu, \mu^2 - \lambda\nu) + \nu \,.\, F_2(\lambda^2, \lambda\mu, \mu\nu, \mu^2 - \lambda\nu)$, multiplication by μ allows the equation of S to be obtained at once in the form $S \equiv y \,.\, F_1(x, y, z, s) + zF_2(x, y, z, s) = 0$, where $m' = \frac{1}{2}(p + 1)$ and the generator $\mu = 0$ is known to belong to the residual intersection. In order that the equation as given may not be reducible by λ, it is necessary that some term of F_2 involve the variable ν to the degree $\frac{1}{2}(p - 1)$; consequently, the given curve must here have $q = \frac{1}{2}(p+1)$. Therefore, $m' = \frac{1}{2}(p+1) = q$; and, since $\frac{3}{2}(p+1) - 1 = \frac{3}{2}(p + \frac{1}{3}) = p + \frac{1}{2}(p + 1) = p + q = m'$, the single generator $\mu = 0$ constitutes the entire residual intersection. This case may be classified thus:

$$\text{III}'.\quad \phi \equiv \lambda F_1 + \nu F_2,\quad p = 2q - 1,\quad m' = q,\quad f_1 = f_2 = 0,\quad \bar{g}_1 = 1,\quad \omega \equiv \mu,$$

if $\bar{g}_1$ refer to the particular generator $\mu = 0$. Geometrically, this case includes only those curves which have $q - 1$ pairs of branches along the double line of Σ and an extra branch in the second sheet which meets the double line at the point $\mu = 0$; hence, any curve ϕ, which has pairs of branches without superfluous branches wherever it meets the double line, save that at one point of the double line occurs a superfluous or single branch lying in the second sheet in that neighborhood, can be cut from Σ by a surface S of order q, which surface is found by the method given, after first making such a change of coördinates, if necessary, as shall bring the equation of the generator at the point in question into the form $\mu = 0$. Clearly, this point cannot lie at the pinch-point.

Still another special case can occur here. If ϕ be written in the form

$$\phi \equiv V_p + \lambda \,.\, V_{p-1} + \lambda^2.\, V_{p-2} + \ldots\ldots + \lambda^\kappa.\, V_{p-\kappa} + \ldots\ldots + \lambda^{p-1}.\, V_1 + \lambda^p.\, V_0 = 0,$$

it has been already seen that $V_p \equiv \mu^{p-q}.\, \overline{V_q}$, where $\overline{V_q} = 0$ gives the points where the curve ϕ meets the double line in the second sheet of Σ. Hence, the equation $\overline{\overline{V_q}} = 0$, where $\overline{\overline{V_q}}$ denotes what $\overline{V_q}$ becomes when μ and ν have been replaced by λ and $-\mu$ respectively therein, gives the q generators meeting the

double line in the first sheet in the points where the (p, q) meets that line in the second sheet. Here will $\overline{V}_q . \overline{\overline{\overline{V}}}_q \equiv F(\lambda\mu, \mu\nu, \mu^2 - \lambda\nu)$, so that $\omega \equiv \lambda^{p-q}.\overline{\overline{\overline{V}}}_q$ gives, by the usual method of substitution, $\omega . V_p \equiv y^{p-q}. F(y, z, s)$; if $p > q + 1$, the required substitution can be performed throughout every other group of terms of the equation $\omega\phi = 0$ and the equation of the surface S is at once obtained. In this case $m' = p$, and the residual intersection is entirely made up of the double line of Σ occurring $p - q$ times in the first sheet and $p - q + \theta_1'$ times in the second sheet of Σ, together with the $q - \theta_1'$ generators in the first sheet of Σ introduced by $\overline{\overline{\overline{V}}}_q = 0$; here θ_1' denotes the number of times λ occurs as a factor in all the terms of $\overline{\overline{\overline{V}}}_q = 0$, and is, geometrically, the order of multiplicity on the curve ϕ of the pinch-point regarded as lying in the second sheet. But the substitution can readily be performed by the usual method in the general case if $\omega \equiv \lambda^2 . \overline{\overline{\overline{V}}}_{q-\theta_1'}$, when $p + q - \theta_1'$ is even, or if $\omega \equiv \lambda^2 . (a\lambda + b\mu) . \overline{\overline{\overline{V}}}_{q-\theta_1'}$, when $p + q - \theta_1'$ is odd, provided in both cases that $\theta_1' \leqq p - q$; here

$$m' = \tfrac{1}{2}(p + q - \theta_1') + 1 \text{ and } \tfrac{1}{2}(p + q - \theta_1' + 1) + 1$$

respectively; that these values be lower than those given in the general case for m', it is necessary that either $\theta_1' > q$ or $\theta_1' > p - q$, neither of which conditions is possible of fulfillment. However, if $p \doteq q$, a surface S of lower order than that given in the general case III. is found, provided that either $V_{p-1} = 0$ or that $\lambda V_{p-1} \overline{\overline{\overline{V}}}_p \equiv F(\lambda^2, \lambda\mu, \mu\nu, \mu^2 - \lambda\nu)$; for multiplying by $\omega \equiv \overline{\overline{\overline{V}}}_p$ in such case makes the substitution possible in every group of terms and gives the resulting surface S, characterized by $m' = p$, $f_1 = f_2 = 0$, $g_1 = p$, $g_2 = 0$; that $\theta_1' = 0$, and that, accordingly, the double line cannot occur in the residual intersection is clear, since the condition that $p = q$ demands that $\nu^q (= \nu^p)$ occur in some term of $\phi = 0$, and such a term can be found only in the first group V_p of that equation; this means that no curve (p, q), where $p = q$, can pass through the pinch-point of Σ, for the presence of a term containing ν^p in the equation of the curve forbids that the curve pass through the pinch-point in the second sheet, and the fact that $p - q = 0$ shows that the curve has no points at all on the double line in the first sheet. Geometrically, the multiplication of $\phi = 0$ by $\overline{\overline{\overline{V}}}_q$ introduces into the locus $\omega\phi = 0$ the generator at every point in the first sheet where the curve $\phi = 0$ meets that line in the second sheet; consequently, the complete intersection of S and Σ has a pair of of branches on the double line wherever that line is met by the curve $\phi = 0$ in

the second sheet; and, if $p=q$, that intersection meets the double line only in the q points, each one of which is a point of multiplicity 2ρ (ρ being a positive integer) on the complete intersection in question. The vanishing of V_{p-1} means that the conic, whose equation is obtained by equating any one of the p factors $a\mu+b\nu$ of V_p to zero, meets the curve ϕ in two consecutive points on the double line of Σ; or, in other words, the curve ϕ has in this case the direction of the conic $a\mu+b\nu=0$ at every point $\lambda=0$, $\mu/\nu=-b/a$, where the curve meets the double line. Since b can never take the value zero here, this condition demands that the curve shall not have the direction of the double line at any point where it meets that line. The alternative condition for V_{p-1}, viz., that $\lambda\,.\,V_{p-1}\,.\,\overline{\overline{V}}_p\equiv F(\lambda^2,\lambda\mu,\mu\nu,\mu^2-\lambda\nu)$, since V_{p-1} does not involve λ, and since $\overline{\overline{V}}_p$ does not contain λ as a factor, requires V_{p-1} to contain $p-1$ of the p factors which make up V_p. Therefore, at $p-1$ of the p points where the curve ϕ meets the double line in the second sheet must it be tangent to a conic whose equation is of the form $a\mu+b\nu=0$, where $b\neq 0$; and it is not difficult to see that the tangent at the remaining point cannot be the double line itself. Therefore, in this case also, the curve ϕ cannot have the direction of the double line at any of the points where it meets that line. This very special case may be characterized thus:

$$\text{III}''.\quad \phi\equiv V_p+\lambda\,.\,V_{p-1}+\lambda^2.\,W_{p-2},\quad \lambda\,.\,V_{p-1}\,.\,\overline{\overline{V}}_p\;F\equiv(\lambda^2,\lambda\mu,\mu\nu,\mu^2-\lambda\nu),$$
$$p=q,\quad m'=p,\quad f_1=f_2=0,\quad \overline{g_1}=p,\;\omega\equiv V_p,$$

where $\overline{g_1}$ refers to the particular generators in the first sheet given by the equation $\overline{\overline{V}}_p=0$.*

Collecting the results thus far obtained for the order of the surface S and the nature and composition of the residual intersection—

1). *When S is a proper surface of any order whatever not less than $\tfrac{1}{2}p+1$, cutting the curve (p,q) from Σ,*

$$m'=\tfrac{1}{2}(p+\tau)+1,\quad f_1=\tfrac{1}{2}(p+\tau)-q+1,\quad f_2=\theta_1+2,\quad g_1=\tau-\theta_1.$$

2). *When S is the surface, or one of the surfaces, of the lowest possible order cutting the curve (p,q) from Σ,*

* The case where $\phi\equiv F_1(\lambda^2,\lambda\mu,\mu\nu,\mu^2-\lambda\nu)+\lambda^2.W_{p-2}=0$, since λ cannot be a factor of all the terms, can be written $\phi\equiv F_1(\mu\nu,\mu^2-\lambda\nu)+\lambda^2.\,W_{p-2}=0$, and presents no new result. For, in order that the required substitution be at once possible, the last group of terms must have $q\leqq\tfrac{1}{2}p-1$, or else the case must reduce itself to that of complete intersection; but the first group demands that $q=\tfrac{1}{2}p$; consequently, this case offers nothing new.

I. $p \geqq 2q$, p *even*, $m' = \frac{1}{2}p + 1$, $f_1 = \frac{1}{2}p - q + 1$, $f_2 = 2$,
$g_1 = 0$, from $\omega \equiv \lambda^2$;

II. $p \geqq 2q$, p *odd*, $m' = \frac{1}{2}(p+3)$, $f_1 = \frac{1}{2}(p+3) - q$, $f_2 = 2 + \theta_1$,
$g_1 = 1 - \theta_1$, from $\omega \equiv \lambda^2 . \omega_1$;

III. $p \leqq 2q$, $m' = q + 1$, $f_1 = 1$, $f_2 = 2$, $g_1 = 2q - p$, from $\omega = \lambda^2 . \omega_{2q-p}$;

unless a surface S of still lower order can be found from the occurrence of one of the following special cases:

I'. $p = 2q$, $\phi \equiv F(\lambda^2, \lambda\mu, \mu\nu, \mu^2 - \lambda\nu)$, $m' = \frac{1}{2}p$, $f = g = 0$, $\omega \equiv 1$.

II'. $p = 2q + 1$, $\phi \equiv \lambda . F_1(\lambda^2, \lambda\mu, \mu\nu, \mu^2 - \lambda\nu) + \mu . F_2(\text{id.})$,
$m' = \frac{1}{2}(p+1)$, $f_1 = f_2 = 1$, $g_1 = 0$, $\overline{g_1} = 0$, from $\omega \equiv \lambda$.

III'. $p = 2q - 1$, $\phi \equiv \lambda . F_1(\lambda^2, \mu\nu, \mu^2 - \lambda\nu) + \nu . F_2(\text{id.})$, $m' = q$,
$f_1 = f_2 = 0$, $g_1 = 0$, $\overline{g_1} = 1$, from $\omega \equiv \mu$.

III''. $p = q$, $\phi \equiv V_p + \lambda . V_{p-1} + \lambda^2 . W_{p-2}$,
$\lambda . \overline{\overline{V_{p-1}}} . \overline{\overline{V_p}} \equiv F(\lambda^2, \lambda\mu, \mu\nu, \mu^2 - \lambda\nu)$, $m' = p$, $f_1 = f_2 = 0$, $\overline{g_1} = p$, from $\omega \equiv \overline{\overline{V_p}}$.

g_1 here refers to generators which may be chosen, in general, at will, and can be wholly or in part taken over into f_2 as g_2; while $\overline{g_1}$ refers to generators which are determined by the given curve and can none of them be transferred to f_2 or otherwise changed in any way without raising the order of S. This distinction between g_1 and $\overline{g_1}$ will be heeded in the farther use of the terms.

Under 1) are included cases I., II., and III., but not the special cases I.', II.', III.', and III.''.

It is easy to find, in any of the above cases, the relation, as regards contact, of the surfaces S and Σ along the double line in either sheet of the latter. Thus it is clear that three cases arise, viz.:

a). If $f_1 = f_2$, then $f_1' = f_2' = f_1 = f_2$, and $f_1'' = f_2'' = 0$.
b). If $f_1 > f_2$, then $f_1' = f_2' = f_2$, $f_1'' = f_1 - f_2$, and $f_2'' = 0$.
c). If $f_1 < f_2$, then $f_1' = f_2' = f_1$, $f_2'' = f_2 - f_1$, and $f_1'' = 0$.

The only curves on Σ which do not meet the generators at all and, consequently, are of the species $(p, 0)$ are the generators themselves. Since these are not proper curves for any value of p greater than unity, all curves (p, q), having $q = 0$ and $p > 1$, will be omitted in the subsequent treatment of the curves on Σ.

The following Table 1 gives, in accordance with the preceding considerations, the possible cases of surfaces S of the lowest possible orders for curves of orders 1–12 on Σ. It will be noticed that each general case, where $p = 2q$, may be

regarded as belonging to both Case I. and Case III.; each $(2q, q)$ is thus entered in the table.

TABLE 1.

m	Species of Curve	Case	m'	f_1'/f_2'	f_1''	f_2''	g_1	$\bar{g}_1$	θ_1
1	(1, 0)	II″.	1	1	0	0	0	0	0
2	(1, 1)	III.	2	1	0	1	1	0	0
		III′.	1	0	0	0	0	1	0
		III″.	1	0	0	0	0	1	0
3	(2, 1)	I., III.	2	1	0	1	0	0	0
		I′.	1	0	0	0	0	0	0
4	(3, 1)	II.	3	2	0	0	1	0	0
		II′.	2	1	0	0	0	0	0
	(2, 2)	III.	3	1	0	1	2	0	0
		III′.	2	0	0	0	0	2	0
5	(4, 1)	I.	3	2	0	0	0	0	0
	(3, 2)	III.	3	1	0	1	1	0	0
		III′.	2	0	0	0	0	1	0
6	(5, 1)	II.	4	2	1	0	1	0	0
				3	0	0	0	0	1
	(4, 2)	I., III.	3	1	0	1	0	0	0
		I′.	2	0	0	0	0	0	0
	(3, 3)	III.	4	1	0	1	3	0	0
		III″.	3	0	0	0	0	3	0
7	(6, 1)	I.	4	2	1	0	0	0	0
	(5, 2)	II.	4	2	0	0	1	0	0
		II′.	3	1	0	0	0	0	0
	(4, 3)	III.	4	1	0	1	2	0	0
8	(7, 1)	II.	5	2	2	0	1	0	0
				3	1	0	0	0	1
	(6, 2)	I.	4	2	0	0	0	0	0
	(5, 3)	III.	4	1	0	1	1	0	0
		III′.	3	0	0	0	0	1	0
	(4, 4)	III.	5	1	0	1	4	0	0
		III″.	4	0	0	0	0	4	0
9	(8, 1)	I.	5	2	2	0	0	0	0
	(7, 2)	II.	5	2	1	0	1	0	0
				3	0	0	0	0	1
	(6, 3)	I., III.	4	1	0	1	0	0	0
		I′.	3	0	0	0	0	0	0
	(5, 4)	III.	5	1	0	1	3	0	0
10	(9, 1)	II.	6	2	3	0	1	0	0
				3	2	0	0	0	1
	(8, 2)	I.	5	2	1	0	0	0	0
	(7, 3)	II.	5	2	0	0	1	0	0
		II′.	4	1	0	0	0	0	0
	(6, 4)	III.	5	1	0	1	2	0	0
	(5, 5)	III.	6	1	0	1	5	0	0
		III″.	5	0	0	0	0	5	0
11	(10, 1)	I.	6	2	3	0	0	0	0
	(9, 2)	II.	6	2	2	0	1	0	0
				3	1	0	0	0	1
	(8, 3)	I.	5	2	0	0	0	0	0
	(7, 4)	III.	5	1	0	1	1	0	0
		III′.	4	0	0	0	0	1	0
	(6, 5)	III.	6	1	0	1	4	0	0
12	(11, 1)	II.	7	2	4	0	1	0	0
				3	3	0	0	0	1
	(10, 2)	I.	6	2	2	0	0	0	0
	(9, 3)	II.	6	2	1	0	1	0	0
				3	0	0	0	0	1
	(8, 4)	I., III.	5	1	0	1	0	0	0
		I′.	4	0	0	0	0	0	0
	(7, 5)	III.	6	1	0	1	3	0	0
	(6, 6)	III.	7	1	0	1	6	0	0
		III″.	6	0	0	0	0	6	0
13	(12, 1)	I.	7	2	4	0	0	0	0
	(11, 2)	II.	7	2	3	0	1	0	0
				3	2	0	0	0	1
	(10, 3)	I.	6	2	1	0	0	0	0
	(9, 4)	II.	6	2	0	0	1	0	0
		II′.	5	1	0	0	0	0	0
	(8, 5)	III.	6	1	0	1	2	0	0

Since it is only in the cases I'., III'., and III''. that a surface S is found which does not have the double line of Σ for a part of its intersection with that surface, it is clear that only in those three cases are restricted systems of equations for the curves in question given by the methods here followed. In the case I'., the intersection is complete; consequently, $S = 0$ and $\Sigma = 0$ form the simplest possible restricted system of equations for the representation of the curve. In the case of a curve coming under III'., a restricted system is formed by $S_1 = 0$, $S_2 = 0$ and $\Sigma = 0$, where S_1 is the surface defined by III'. and S_2 is any one of those defined by the general case with the value $\frac{1}{2}(p+1)$ inserted for q and the condition imposed that g_1 shall not contain the generator given by $\mu = 0$; thus the surface S_1 will be characterized by having

$$m' = p, \quad f_1 = f_2 = 0, \quad \text{and } \overline{g_1} = 1;$$

while S_2 will have

$$m' = \tfrac{1}{2}(p+\tau)+1, \quad \text{and } f_1 = \tfrac{1}{2}(\tau+1), \quad f_2 = \theta_1 + 2, \quad g_1 = \tau - \theta_1 \text{ and } \theta_2 = 0.$$

In the simplest case, where $\tau = 2q - p$ and $\theta_1 = 0$, the surface S_2 is defined like the S of Case III. If the curve fall under Case III''., a restricted system is given by $S_1 = 0$, $S_2 = 0$ and $\Sigma = 0$, where S_1 is defined by Case III''. and S_2 is any one of those defined by the general case by putting $p = q$ therein and insuring, by choice of ω_τ, that g_1 contain none of the generators given by $\overline{V_p} = 0$ and represented by $\overline{g_1}$ in III''.; thus S_1 will have

$$m' = p, \quad f_1 = f_2 = 0, \quad \overline{g_1} = p;$$

and S_2 will be characterized by

$$m' = \tfrac{1}{2}(p+\tau)+1, \quad f_1 = \tfrac{1}{2}(\tau - p)+1, \quad f_2 = \theta_1 + 2, \quad g_1 = \tau - \theta_1,$$

which, for the lowest value, p, that τ can take, gives an S_2 having

$$m' = p+1, \quad f_1 = 1, \quad f_2 = 2, \quad g_1 = \tau.$$

It is now possible to determine how many species of proper curves (p, q) may result from the intersection of surfaces S of any given order m' with Σ, when it is required that S be a surface of the lowest possible order thus containing the curve, and hence having its residual intersection made up entirely of the double line and generators of Σ. If the intersection be of such nature as to

give any one of the eight cases,

1). $f_1 = f_2 = g_1 = g_1 = 0$;
2). $f_1 = f_2 = g_1 = 0$, $\overline{g_1} = 1$, when $m' \geqq 2$;
3). $f_1 = f_2 = g_1 = 0$, $\overline{g_1} = m'$;
4). $f_1 = f_2 = 1$, $g_1 = \overline{g_1} = 0$;
5). $f_1 = 1$, $f_2 = 2$, $g_1 = \rho$, $\overline{g_1} = 0$, where $2 \leqq \rho \leqq m' - 1$;
6). $f_1 = \rho$, $f_2 = 2$, $g_1 = \overline{g_1} = 0$, where $1 \leqq \rho \leqq m' - 1$;
7). $f_1 = \rho$, $f_2 = 2$, $g_1 = 1$, $\overline{g_1} = 0$, where $1 \leqq \rho \leqq m' - 1$;
8). $f_1 = \rho$, $f_2 = 3$, $g_1 = \overline{g_1} = 0$, where $3 \leqq \rho \leqq m' - 1$;

and if the residual of the intersection is a proper curve, then is S, in general, a surface of the lowest possible order that can be passed through the residual curve, Σ being excepted when $m' \geqq 4$. The lower limit of the order of the curves (p, q) here occurring is evidently $m = 2(m' - 1)$, and the species of the curve is given by the following formulas, which are arranged and numbered to agree with the cases of page 206:

I. $f_1 = \rho$, $f_2 = 2$, $g_1 = \overline{g_1} = 0$, $p = 2(m' - 1)$, $q = m' - \rho$,
$1 \leqq \rho \leqq m' - 1$, $p \geqq 2q$, and *even*; from 6) above.

II. $f_1 = \rho$, $f_2 = 2$, $g_1 = 1$, $\overline{g_1} = 0$, $p = 2m' - 3$, $q = m' - \rho$,
$2 \leqq \rho \leqq m' - 1$, $p \geqq 2q$ and *odd*, $\theta_1 = 0$; from 7) above.
$f_1 = \rho$, $f_2 = 3$, $g_1 = \overline{g_1} = 0$, $p = 2m' - 3$, $q = m' - \rho$,
$3 \leqq \rho \leqq m' - 1$, $p \geqq 2q$ and *odd*, $\theta_1 = 1$; from 8) above.

III. $f_1 = 1$, $f_2 = 2$, $g_1 = \rho$, $\overline{g_1} = 0$, $p = 2(m' - 1) - \rho$,
$q = m' - 1$, $1 \leqq \rho \leqq m' - 1$, $p \leqq 2q$; from 5) above.

I′. $f_1 = f_2 = g_1 = \overline{g_1} = 0$, $p = 2m'$, $q = m'$, $p = 2q$; from 1) above.

II′. $f_1 = 1$, $f_2 = 1$, $g_1 = \overline{g_1} = 0$, $p = 2m' - 1$, $q = m' - 1$,
$p = 2q + 1$; from 4) above.

III′. $f_1 = f_2 = 0$, $g_1 = 0$, $\overline{g_1} = 1$, $p = 2m' - 1$, $q = m'$, $p = 2q - 1$;
from 2) above.

III″. $f_1 = f_2 = 0$, $g_1 = 0$, $\overline{g_1} = m'$, $p = m'$, $q = m'$, $p = q$; from 3) above.

Cases III′. and III″. become identical when $m' = 1$, but for no other values of m' will any repetitions occur from these formulas.

The determination of the number of distinct species of curves, as well as

the number of their different orders, which can be cut from Σ by surfaces of any order m' under the given conditions, can now be made. Arranged according to the above cases, the number of distinct species of curve for any value of m' comes out thus:

I. $p=2m'-2$, $q=m'-1$, $m'-2$, ..., 1, making $m'-1$ species.
II. $p=2m'-3$, $q=m'-2$, $m'-3$, ..., 1, making $m'-2$ species.
III. $p=2m'-3$, $2m'-4$, ..., $m'-1$, $q=m'-1$, making $m'-1$ species.
I′. $p=2m'$, $q=m'$, a single species.
II′. $p=2m'-1$, $q=m'-1$, a single species.
III′. $p=2m'-1$, $q=m'$, a single species.
III″. $p=m'$, $q=m'$, a single species.

Consequently, all together, curves (p, q) of

$$m'-1+m'-2+m'-1+4=3m'$$

distinct species are found. Of different orders, all from $2m'-2$ to $3m'$ occur, making a total of $m'+3$, so long as $m' \geqq 2$; when $m'=1$, this result must be diminished by unity, since no curve of the lowest order, $2m'-2$, then exists. Hence, the following

THEOREM: *Surfaces S of any given order $m' \geqq 2$, can cut from Σ curves (p, q) of $3m'$ distinct species and of $m'+3$ different orders, where S is a surface of the lowest possible order containing the curve (except Σ, if $m' \geqq 4$) and, therefore, has its residual intersection with Σ made up entirely of generators of the latter surface.*

If the curve (p, q) have $q=1$, a plane through the double line of Σ will contain only one point of the curve not lying on that line. Such a curve is unicursal; the surfaces S, of order $m' \geqq 2$, can then, under the given conditions, cut from Σ two unicursal curves for each value of m'; these two curves will be of the species $(2m'-2, 1)$ and $(2m'-3, 1)$ and of the orders $2m'-1$ and $2m'-2$ respectively. When $m'=1$, the unicursal curves found in the same way are of the species $(1, 0)$ and $(1, 1)$ and of the orders 1 and 2 respectively

Results for the lower values of m', in accordance with the formulas stated, are given in Table 2. In this table, as in some of the preceding formulas, curves (p, q), where $p=2q$, which might be given under both Case I. and Case III., are inserted as occurring only in Case I.

TABLE 2.

m'	Case	Species of Curve	m	f_1' f_2'	f_1''	f_2''	g_1	$\overline{g_1}$	θ_1
1	I'.	(2, 1)	3	0	0	0	0	0	0
	II'.	(1, 0)	1	1	0	0	0	0	0
	III'. III''.	(1, 1)	2	0	0	0	0	1	0
2	I.	(2, 1)	3	1	0	1	0	0	0
	III.	(1, 1)	2	1	0	1	1	0	0
	I'.	(4, 2)	6	0	0	0	0	0	0
	II'.	(3, 1)	4	1	0	0	0	0	0
	III'	(3, 2)	5	0	0	0	0	1	0
	III''.	(2, 2)	4	0	0	0	0	2	0
3	I.	(4, 2)	6	1	0	1	0	0	0
		(4, 1)	5	2	0	0	0	0	0
	II.	(3, 1)	4	2	0	0	1	0	0
	III.	(3, 2)	5	1	0	1	1	0	0
		(2, 2)	4	1	0	1	2	0	0
	I'.	(6, 3)	9	0	0	0	0	0	0
	II'.	(5, 2)	7	1	0	0	0	0	0
	III'.	(5, 3)	8	0	0	0	0	1	0
	III''.	(3, 3)	6	0	0	0	0	3	0
4	I.	(6, 3)	9	1	0	1	0	0	0
		(6, 2)	8	2	0	0	0	0	0
		(6, 1)	7	2	1	0	0	0	0
	II.	(5, 2)	7	2	0	0	1	0	0
		(5, 1)	6	2	1	0	1	0	0
				3	0	0	0	0	1
	III.	(5, 3)	8	1	0	1	1	0	0
		(4, 3)	7	1	0	1	2	0	0
		(3, 3)	6	1	0	1	3	0	0
	I'.	(8, 4)	12	0	0	0	0	0	0
	II'.	(7, 3)	10	1	0	0	0	0	0
	III'.	(7, 4)	11	0	0	0	0	1	0
	III''.	(4, 4)	8	0	0	0	0	4	0
5	I.	(8, 4)	12	1	0	1	0	0	0
		(8, 3)	11	2	0	0	0	0	0
		(8, 2)	10	2	1	0	0	0	0
		(8, 1)	9	2	2	0	0	0	0
	II.	(7, 3)	10	2	0	0	1	0	0
		(7, 2)	9	2	1	0	1	0	0
				3	0	0	0	0	1
		(7, 1)	8	2	2	0	1	0	0
				3	1	0	0	0	1
	III.	(7, 4)	11	1	0	1	1	0	0
		(6, 4)	10	1	0	1	2	0	0
		(5, 4)	9	1	0	1	3	0	0
		(4, 4)	8	1	0	1	4	0	0
	I'.	(10, 5)	15	0	0	0	0	0	0
	II'.	(9, 4)	13	1	0	0	0	0	0
	III'.	(9, 5)	14	0	0	0	0	1	0
	III''.	(5, 5)	10	0	0	0	0	5	0
6	I.	(10, 5)	15	1	0	1	0	0	0
		(10, 4)	14	2	0	0	0	0	0
		(10, 3)	13	2	1	0	0	0	0
		(10, 2)	12	2	2	0	0	0	0
		(10, 1)	11	2	3	0	0	0	0
	II.	(9, 4)	13	2	0	0	1	0	0
		(9, 3)	12	2	1	0	1	0	0
				3	0	0	0	0	1
		(9, 2)	11	2	2	0	1	0	0
				3	1	0	0	0	1
		(9, 1)	10	2	3	0	1	0	0
				3	2	0	0	0	1
	III.	(9, 5)	14	1	0	1	1	0	0
		(8, 5)	13	1	0	1	2	0	0
		(7, 5)	12	1	0	1	3	0	0
		(6, 5)	11	1	0	1	4	0	0
		(5, 5)	10	1	0	1	5	0	0
	I'.	(12, 6)	18	0	0	0	0	0	0
	II'.	(11, 5)	16	1	0	0	0	0	0
	III'.	(11, 6)	17	0	0	0	0	1	0
	III''.	(6, 6)	12	0	0	0	0	6	0
7	I.	(12, 6)	18	1	0	1	0	0	0

V.—*Singularities of the Curves* (p, q) *on* Σ *in Terms of* p *and* q.

It is now possible to find the singularities of any given curve on Σ by making use of the knowledge of the nature of the residual intersection of Σ with the surface S, as that surface has been defined in the preceding pages.

Let the characteristics of the developable surface, having the curve in question for its edge of regression, and of the cone standing upon that curve be denoted, in accordance with the usage of Cayley and Salmon,* by the letters $m, n, r, \alpha, \beta, x, y, g, h$ and H.

As already seen, $m = p + q$. It is, in general, possible that the curve of intersection of S and Σ shall contain no cusps arising from the occurrence of stationary contact between these two surfaces; hence, for one such curve, at least, it may be assumed that β has its lowest value, zero. For the third singularity of the given curve, the rank, r, can be found in the following manner:

Given two surfaces, whose equations are $S = 0$ and $\Sigma = 0$, the condition that a tangent to their curve of intersection meet the arbitrary line represented by the equations

$$a_1x + b_1y + c_1z + d_1s = 0 \text{ and } a_2x + b_2y + c_2z + d_2s = 0 \text{ is that}$$

$$\begin{vmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ \frac{\partial S}{\partial x} & \frac{\partial S}{\partial y} & \frac{\partial S}{\partial z} & \frac{\partial S}{\partial s} \\ \frac{\partial \Sigma}{\partial x} & \frac{\partial \Sigma}{\partial y} & \frac{\partial \Sigma}{\partial z} & \frac{\partial \Sigma}{\partial s} \end{vmatrix} = 0,$$

which may be denoted by $\Delta = 0$, where the surface Δ represents the locus of points, the intersections of whose polar planes with respect to S and Σ meet the arbitrary line. If the curve in question be the complete intersection of S and Σ, the rank desired will be the number of points common to S, Σ and Δ; i. e., the product of the orders of those three surfaces, subject to a reduction for the multiple points of the curve. But, if the curve be taken as the partial intersection of S and Σ, a further reduction of that product is necessary in order to obtain r.†

The surface S has been defined, in the general case, when so determined as

---

* Salmon, "Geometry of Three Dimensions," pp. 291–293.

† Salmon, "Geometry of Three Dimensions," p. 308.

to contain the curve (p, q) and have the residual of its intersection with Σ made up entirely of straight lines, by the formulas (cf. p. 206),

$$m' = \tfrac{1}{2}(p+\tau)+1, \quad f_1 = \tfrac{1}{2}(p+\tau) - q + 1, \quad f_2 = 2, \quad g_1 = \tau, \text{ when } \theta_1 = 0.$$

The surface Σ is known to be of order three and to have a double line. Therefore, if M', F_1, F_2 and G_1 represent, in regard to Δ and Σ, what m', f_1, f_2, and g_1 respectively, denote with reference to S and Σ, it is clear that the surface Δ will have

$$M' = \tfrac{1}{2}(p+\tau)+2, \quad F_1 = \tfrac{1}{2}(p+\tau) - q + 1, \quad F_2 = 2$$

in all cases. As for G_1, it is evident that its value depends on the choice of ω_τ; thus, if ω_τ be so chosen as to represent τ different generators in the first sheet of Σ, it follows that $G_1 = 0$, while if ω_τ include any such generator more than once, then will that generator occur on Δ, so that in such case $G_1 > 0$. In general, if all generators entering more than once in ω_τ be ρ in number and contribute a total order of γ as their component of g_1, where $2\rho \leqq \gamma \leqq \tau$, then will Δ contain ρ generators of Σ and have $G_1 = \gamma - \rho$; for any generator in the first sheet of Σ, which occurs ε times in the intersection of S and Σ, will occur $\varepsilon - 1$ times in the intersection of Δ and Σ.

The curve (p, q) meets the surface Δ, in general, in

$$m \,.\, M' = (p+q)[\tfrac{1}{2}(p+\tau)+2]$$

points, which number must be subjected to reductions in two ways in order to obtain r. First, since the lines common to Δ and Σ are multiple to an order at least as great as two on one or both the surfaces S and Σ, it follows that the polar plane or planes with respect to S and Σ for all points on these lines are one or both indefinite; consequently, the points of intersection of the curve (p, q) with such multiple lines of Δ should be rejected from the above number; this demands a reduction, since (p, q) has $p - q$ points on the double line in the first sheet of Σ, q points on that line in the second sheet of Σ, and q points on every generator in the first sheet of Σ, amounting to

$$(p-q)\,.\,F_1 + qF_2 + qG_1 = (p-q)[\tfrac{1}{2}(p+\tau) - q + 1] + 2q + q\,.\,(\gamma - \rho).$$

Secondly, wherever the curve (p, q) meets a line common to S and Σ, these two surfaces have contact, since an element of the curve and an element of the common line determine there a plane tangent to a sheet of either surface. Every such plane meets the arbitrary line in question, but such points must be rejected

from the above number, since, in general, the tangent to the curve there will not intersect the arbitrary line. This demands a reduction by p for the points where the curve meets the double line, by ρq for the points on the ρ generators occurring on Δ, and by $(\tau - \gamma) . q$ for the points on the $\tau - \gamma$ generators occurring singly in the residual intersection of S and Σ. Therefore, the second reduction amounts to

$$p + (\rho + \tau - \gamma) . q.$$

Hence, if no further condition of contact between S and Σ be imposed, it is obtained that

$$\begin{aligned} r &= m . M' - (p - q) . F_1 - q . F_2 - q . G_1 - p - (\rho + \tau - \gamma) . q \\ &= p . (M' - F_1 - 1) + q . (M' + F_1 - F_2 - \tau) \\ &= q . (2p - q + 1). \end{aligned}$$

And if it be further required that S and Σ have ordinary contact at H points and stationary contact at β points, the formula for the rank of the given curve (p, q) is

$$r = q . (2p - q + 1) - 2H - 3\beta .$$

Double points resulting from the intersection on the double line of Σ of a pair of branches of the curve (p, q), the branches lying one in either sheet in that neighborhood, are among the singularities included in h and not in H; for, while they are actual double points of the curve, regarded by itself, they are only apparent double points, regarded from the point of view of the geometry on the surface Σ. If such apparent-actual double points have their number denoted by h_2, where $h = h_1 + h_2$, it is evident that $h_2 \leqq p/2$; h_1 will then refer to points which are, from the consideration of the geometry on the surface Σ as well as from the point of view of the curve itself, apparent double points; and H will refer only to those multiple points arising from imposed contact of S with Σ, i. e., actual double points from both points of view. Similarly, a cusp, from the point of view of the curve, arises when two branches, lying one in either sheet in that neighborhood, intersect at the pinch-point; but, regarded from the standpoint of the geometry on the surface Σ, such a singularity is not a cusp and consequently will not be included in β, which represents the number of cusps resulting from imposed stationary contact between S and Σ, i. e., singularities which are cusps from both points of view; such apparent-actual cusps will be included in h_2.

The ordinary Plückerian equations connect the singularities here considered by the formulas

$$\begin{aligned} h &= \tfrac{1}{2}\left[m(m-1)-r\right], & x &= \tfrac{1}{2}\left[r(r-1)-3m-n\right], \\ n &= 3(r-m)+\beta, & y &= x-(n-m), \\ \alpha &= 2(n-m)+\beta, & g &= \tfrac{1}{2}\left[n(n-1)-r-3\alpha\right]. \end{aligned}$$

These equations, with the help of the values of m and r already found, give the following complete set of formulas for the singularities of the curve (p, q) in terms of p, q, H, and β:

$$\begin{aligned} m &= p+q, \\ h &= \tfrac{1}{2}p(p-1)+q(q-1), \\ n &= 3(2pq-q^2-p)-2(3H+4\beta), \\ \alpha &= 4p(3q-2)-2q(3q+1)-3(4H+5\beta), \\ r &= q(2p-q+1)-2H-3\beta, \\ x &= 2pq(pq-q^2+q-2)+\tfrac{1}{2}q(q^3-2q^2+5q-4) \\ &\qquad -\tfrac{1}{2}(4pq-2q^2+2q-1)(2H+3\beta)+\tfrac{1}{2}(2H+3\beta)^2+3H+4\beta, \\ y &= 2pq(pq-q^2+q-5)+\tfrac{1}{2}q(q^3-2q^2+11q-2) \\ &\qquad +4p-\tfrac{1}{2}(4pq-2q^2+2q-1)(2H+3\beta)+\tfrac{1}{2}(2H+3\beta)^2+3(3H+4\beta), \\ g &= \tfrac{9}{2}(2pq-q^2-p)^2-11q(2p-q)+\tfrac{1}{2}(27p+5q) \\ &\qquad -\left[6(2pq-q^2-p)-7\right](3H+4\beta)+2(3H+4\beta)^2+H. \end{aligned}$$

If the deficiency of the curve (p, q) be denoted by D, it will be shown later (cf. page 220) that

$$D = p(q-1)-\tfrac{1}{2}q(q+1)+1-H-\beta.$$

The number of varieties of curve of each species (p, q), according to the possible conditions of ordinary and stationary contact between the surfaces S and Σ, is evidently $\tfrac{1}{2}(D+1)(D+2)$.

It was seen on page 210 that the curves (p, q) having $q = 1$, are unicursal; the equation for D above shows that such curves have the deficiency zero, and gives as the condition for zero deficiency when $H = \beta = 0$ that q have the value unity.

The singularities of several curves of the lower orders, as computed by the help of the formulas found above, are presented in Table 3.

TABLE 3.

m	Species.	h	D	H	β	r	n	a	x	y	g
1	(1, 0)	0	0	0	0	0	0	0	0	0	0
2	(1, 1)	0	0	0	0	2	0	0	0	0	0
3	(2, 1)	1	0	0	0	4	3	0	0	0	1
4	(3, 1)	3	0	0	0	6	6	4	6	4	6
	(2, 2)	3	0	0	0	6	6	4	6	4	6
5	(4, 1)	6	0	0	0	8	9	8	16	12	20
	(3, 2)	5	1	0	0	10	15	20	30	20	70
			0	1	0	8	9	8	16	12	20
			0	0	1	7	7	5	10	8	10
6	(5, 1)	10	0	0	0	10	12	12	30	24	43
	(4, 2)	8	2	0	0	14	24	36	70	52	125
			1	1	0	12	18	24	48	36	111
			1	0	1	11	16	21	38	28	83
			0	2	0	10	12	12	30	24	43
			0	1	1	9	10	9	22	18	27
			0	0	2	8	8	6	13	15	13
	(3, 3)	9	1	0	0	12	18	24	48	36	111
			0	1	0	10	12	12	30	24	43
			0	0	1	9	10	9	22	18	27
7	(6, 1)	15	0	0	0	12	15	16	48	40	75
	(5, 2)	12	3	0	0	18	33	52	126	100	441
			2	1	0	16	27	40	96	76	283
			2	0	1	15	25	37	82	64	237
			1	2	0	14	21	28	70	56	162
			1	1	1	13	19	25	58	46	127
			1	0	2	12	17	22	47	37	97
			0	3	0	12	15	16	48	40	75
			0	2	1	11	13	13	38	32	53
			0	1	2	10	11	10	29	36	35
			0	0	3	9	9	7	21	19	31
	(4, 3)	12	3	0	0	18	33	52	126	100	441
			2	1	0, etc., repeating the above cases in order.						
8	(7, 1)	21	0	0	0	14	18	20	70	60	116
	(6, 2)	17	4	0	0	22	42	68	198	164	748
			3	1	0	20	36	56	160	132	536
			3	0	1	19	34	53	142	116	472
			2	2	0	18	30	44	126	104	360

m	Species.	*h*	*D*	*H*	*β*	*r*	*n*	*a*	*x*	*y*	*g*
			2	1	1	17	28	41	110	90	308
			2	0	2	16	26	38	95	77	260
			1	3	0	16	24	32	96	80	220
			1	2	1	15	22	29	82	68	180
			1	1	2	14	20	26	69	57	144
			1	0	3	13	18	23	57	47	112
			0	4	0	14	18	20	70	60	116
			0	3	1	13	16	17	58	50	88
			0	2	2	12	14	14	47	41	64
			0	1	3	11	12	11	37	33	44
			0	0	4	10	10	8	28	26	28
	(5, 3)	16	5	0	0	24	48	80	240	236	1128
			4	1	0	22	42	68	198	164	748
			4	0	1	21	40	65	178	146	672
			3	2	0	20	36	56	160	132	536
			3	1	1	19	34	53	142	116	472
			3	0	2	18	32	50	125	101	412
			2	3	0	18	30	44	126	104	360
			2	2	1	17	28	41	110	90	308
			2	1	2	16	26	38	95	77	260
				0	3	15	24	35	81	65	216
			1	4	0	16	24	32	96	80	220
			1	3	1	15	22	29	82	68	180
			1	2	2	14	20	26	69	57	144
			1	1	3	13	18	23	57	47	112
			1	0	4	12	16	20	46	38	84
			0	5	0	14	18	20	70	60	116
			0	4	1	13	16	17	58	50	88
			0	3	2	12	14	14	47	41	64
			0	2	3	11	12	11	37	33	44
			0	1	4	10	10	8	28	26	28
			0	0	5	9	8	5	20	20	16
	(4, 4)	18	3	0	0	20	36	56	160	132	536
			2	1	0	18	30	44	126	102	310
			2	0	1	17	28	41	110	90	303

VI.—*Geometry on Σ from the Point of View of Plane Curves.*

Any curve (p, q) in the geometry on Σ has as its analogue in plane geometry an entirely definite curve; the equation of the former being $\phi(\lambda, \mu, \nu) = 0$, the equation of the corresponding plane curve is $\phi(x, y, z) = 0$. The order of (p, q) as a curve in space is $p+q$, but the order of the corresponding curve in the plane, and also of the given curve from the point of view of the geometry on Σ, is p. To the $p-q$ points of (p, q), which lie on the double line in the first sheet of Σ corresponds on the plane curve a multiple point, the order of whose multiplicity is $p-q$; the line on Σ is given by $\lambda/\nu = \mu/\nu = 0$, and the point in the plane by $x=0, y=0$. To this fact that to the $p-q$ points on a line on Σ corresponds a $(p-q)$-tuple point in the plane are due the chief differences between the geometry on Σ and plane geometry.

1. *Intersections of Two Curves (p, q) and (p', q').*

Two plane curves of orders p and p' intersect in pp' points. But since the two curves (p, q) and (p', q') will, in general, have no common points on the double line in the first sheet of Σ to correspond to the points of intersection of the two analogous plane curves at the multiple point $x=0, y=0$, the number of intersections of the curves (p, q) and (p', q') will be less than pp' by the number of points of intersection of the two corresponding plane curves at their multiple point in question, i. e, pp' must be diminished by $(p-q)(p'-q')$. Hence, if (p, q) and (p', q') have no points of intersection on the double line in the first sheet of Σ, they will intersect in $pp'-(p-q)(p'-q')$ points. If, however, the two curves (p, q) and (p', q') have a common point or points on the double line in the first sheet of Σ, further consideration is necessary. If the equations of the two curves are

$$\begin{aligned}\phi(\lambda, \mu, \nu) = U_p + \nu \,.\, U_{p-1} + \cdots + \nu^{\kappa} . U_{p-\kappa} + \cdots \\ + \nu^{q-1} . U_{p-q+1} + \nu^{q} . U_{p-q} = 0 \text{ and}\\ \phi'(\lambda, \mu, \nu) = U'_{p'} + \nu \,.\, U'_{p'-1} + \cdots + \nu^{\kappa'} . U'_{p'-\kappa'} + \cdots \\ + \nu^{q'-1} . U'_{p'-q'+1} + \nu^{q'} . U'_{p'-q'} = 0,\end{aligned}$$

the condition that θ points of the line in question be common to the two curves, or that the order of intersection of the two curves on that line be θ, is that U_{p-q} and $U'_{p'-q'}$ have θ linear factors common. If the equations of the two corresponding plane curves be now obtained by replacing λ, μ, and ν by x, y, and z

respectively in the equations of (p, q) and (p', q'), giving $\phi(x, y, z) = 0$ and $\phi'(x, y, z) = 0$, the condition for θ intersections on the double line in the first sheet of Σ in the former case becomes in the latter case the condition that θ branches of the one curve be tangent to θ branches of the other curve at the point $x = 0$, $y = 0$. Hence, the two plane curves will have $pp' - (p - q)(p' - q') - \theta$ intersections apart from the multiple point in question. Similarly, the given curves (p, q) and (p', q') will have $pp' - (p - q)(p' - q') - \theta$ intersections apart from the double line in the first sheet of Σ. This number, together with the θ intersections lying on the double line in the first sheet of Σ, gives a total of $pp' - (p - q)(p' - q')$ intersections in this case also. Therefore, from the point of view of the geometry on Σ is established for all cases the

THEOREM.—*The two curves (p, q) and (p', q') on Σ have $pq' + q(p' - q')$ intersections.** θ of these intersections will lie on the double line in the first sheet of Σ when, and only when, the two corresponding plane curves have θ branches of the one tangent to θ branches of the other at the multiple point $x = 0$, $y = 0$. If either $p = q$ or $p' = q'$, then will not only θ have the value zero, but, furthermore, the number of intersections of (p, q) and (p', q') will be the same as the number of intersections of the two corresponding plane curves.

The above formula includes only those intersections which occur as such in the geometry on the surface Σ and does not take account of cases where a branch or branches of each curve meets the double line of Σ at the same point, the branch or branches of the one curve lying in the one sheet and the branch or branches of the other curve lying in the other sheet of Σ in that neighborhood; but the intersections thus occasioned, although to be regarded as only apparent from the standpoint of the geometry on Σ, are actual from the point of view of the curves (p, q) and (p', q') as curves in space. If the number of such intersections be denoted by θ_0, it is evident that $0 \leqq \theta_0 \leqq (p - q)q' + (p' - q')q$, and that $pq' + p'q - qq' + \theta_0$ is the number of intersections possible to any two curves (p, q) and (p', q') regarded as curves in space; evidently this number can be greater than, equal to, or less than the number of intersections of the corresponding curves in the plane; it will necessarily be the same as in the case of the analogous plane curves when both $p = q$ and $p' = q'$.

* This agrees with the formula $(a_\alpha, b_\beta) = a\beta + b\alpha - 3\alpha\beta$ given by Professor Story, "On the Number of Intersections of Curves Traced on a Scroll of any Order," Johns Hopkins University Circulars, August, 1883.

2. *Double Points on* (p, q).

A plane curve of order p can have at most $\frac{1}{2}(p-1)(p-2)$ double points. A point of the $(p-q)^{\text{th}}$ order of multiplicity counts as equivalent to $\frac{1}{2}(p-q)(p-q-1)$ double points; hence, the plane curve, which is the analogue of the curve (p, q) on Σ, can have a maximum of

$$\tfrac{1}{2}(p-1)(p-2)-\tfrac{1}{2}(p-q)(p-q-1)=p(q-1)-\tfrac{1}{2}q(q+1)+1$$

double points apart from the $(p-q)$-tuple point $x=0$, $y=0$. Consequently, from the point of view of the geometry on Σ, the curve (p, q) can have at most $p(q-1)-\frac{1}{2}q(q+1)+1$ double points; hence, $D=p(q-1)-\frac{1}{2}q(q+1)+1-H-\beta$, as stated on page 215. If a multiple point or multiple points, whose total order of multiplicity is γ, lie on the double line in the first sheet of Σ, then must the branches of the corresponding plane curve have contact of the total order γ at the multiple point $x=0$, $y=0$; evidently, γ must have the value zero when $p=q$; and, furthermore, any curve (p, q), where $p=q$, has the same maximum number of double points as the corresponding plane curve.

The results obtained thus far in this section have to do only with actual double points; i. e., points whose order of multiplicity comes entirely from the intersection of branches lying in the same sheet in the neighborhood of the point in question in each case. But it has been already noticed that apparent-actual multiple points—at most, only apparent from the standpoint of the geometry on Σ, but actual from the point of view of (p, q) as a curve in space—may occur from the intersection on the double line of branches of the curve, some of which branches lie in the one sheet of Σ and another or others lie in the other sheet of Σ in the neighborhood of the point of intersection in question. If no actual multiple points occur on the double line of Σ, there can be at most q such apparent-actual double points on the curve (p, q), if $p \geqq 2q$, and at most $p-q$ such double points on that curve, if $p \leqq 2q$. In general, if β_1 branches intersect at a point of the double line, and all lie in the first sheet of Σ in that neighborhood, and also β_2 branches, all lying in the second sheet of Σ in the neighborhood of the double line, intersect at the same point of that line, the point in question counts as a $\frac{1}{2}(\beta_1+\beta_2)(\beta_1+\beta_2-1)$-tuple point of the curve to which those branches belong, if that curve (p, q) be regarded as a curve in space. The same point has for its order of multiplicity on (p, q), regarded from the standpoint of the geometry on Σ and the corresponding plane curve, only the sum $\frac{1}{2}\beta_1(\beta_1-1)$

$+ \frac{1}{2}\beta_2(\beta_2 - 1)$; hence, the increase in the order of multiplicity of the point, due to the introduction of the apparent-actual multiple points, is

$$\tfrac{1}{2}(\beta_1 + \beta_2)(\beta_1 + \beta_2 - 1) - \tfrac{1}{2}\beta_1(\beta_1 - 1) - \tfrac{1}{2}\beta_2(\beta_2 - 1) = \beta_1\beta_2,$$

which may be as great as $q(p-q)$. Therefore, the maximum sum of the orders of multiplicity of all the multiple points of (p, q), regarded as a curve in space, is

$$p(q-1) - \tfrac{1}{2}q(q+1) + 1 + q(p-q) = p(2q-1) - \tfrac{1}{2}q(3q+1) + 1.$$

So the curve (p, q) can have the sum of the orders of multiplicity of all its multiple points less than, equal to, or greater than the corresponding sum in the case of the analogous plane curve. The above number $p(2q-1) - \frac{1}{2}q(3q+1) + 1$ reduces to $\frac{1}{2}(p-1)(p-2)$, when $p = q$, as is evident geometrically, since no point of the curve (p, q), when $p = q$, lies on the double line in the first sheet of Σ. Any curve (p, q), which has $q = 1$ and, consequently, is unicursal, and, from the point of view of the geometry on Σ, of deficiency zero, can still have an order of multiplicity of all its multiple points together of at most $p - 1$, according as the $p - 1$ points on the double line in the first sheet of Σ lie, in part or even wholly, at the point where that line is met by the curve in question in the second sheet of Σ; thus the quantity $p(2q-1) - \frac{1}{2}q(3q+1) + 1$ reduces to $p - 1$ when q has the value unity.

3. *Determination of the Curve* (p, q).

The equation of the curve (p, q), when written in the form

$$\phi \equiv U_p + \nu \,.\, U_{p-1} + \ldots + \nu^{\kappa}.\, U_{p-\kappa} + \ldots + \nu^{q-1}.\, U_{p-q+1} + \nu^{q}.\, U_{p-q} = 0,$$

is seen to contain, in the general case,

$$p + 1 + p + p - 1 + \ldots + p - q + 2 + p - q + 1$$
$$= pq + p - \tfrac{1}{2}q(q-1) + 1$$

terms; consequently, the curve (p, q) is determined by $p(q+1) - \frac{1}{2}q(q-1)$ points.

The same can be seen also in this way: The general p-thic in the plane is determined by $\frac{1}{2}p(p+3)$ points. The equation above, representing (p, q), has all the terms of the general p-thic save those in the powers of ν beyond ν^q, which are

$$\nu^{q+1}(U_{p-q-1} + \nu \,.\, U_{p-q-2} + \ldots + \nu^{p-q-1}.\, U_0).$$

The number of these terms is

$$p-q+p-q-1+\ldots\ldots+2+1=\tfrac{1}{2}(p-q)(p-q+1).$$

This number of conditions must be subtracted from $\frac{1}{2}p(p+3)$, giving

$$\tfrac{1}{2}p(p+3)-\tfrac{1}{2}(p-q)(p-q+1)=p(q+1)-\tfrac{1}{2}q(q-1)$$

as the number of conditions or points necessary to determine a curve on Σ.

Unicursal curves, having always $q=1$, are each determined by $2p$ points, as the formula shows; and any (p, q), where $p=q$, is seen, from the point of view of its analogue in the plane, as well as from the formula given, to be determined by $\frac{1}{2}p(p+3)$ points.

Table 4 gives the number of points necessary to determine a curve (p, q) of the species indicated, for all curves having $p \leqq 10$.

TABLE 4.

Curve	Points	Curve	Points	Curve	Points	Curve	Points	Curve	Points
(1, 0)	1	(5, 2)	14	(7, 3)	25	(8, 8)	44	(10, 3)	37
(1, 1)	2	(5, 3)	17	(7, 4)	29	(9, 1)	18	(10, 4)	44
(2, 1)	4	(5, 4)	19	(7, 5)	32	(9, 2)	26	(10, 5)	50
(2, 2)	5	(5, 5)	20	(7, 6)	34	(9, 3)	33	(10, 6)	55
(3, 1)	6	(6, 1)	12	(7, 7)	35	(9, 4)	39	(10, 7)	59
(3, 2)	8	(6, 2)	17	(8, 1)	16	(9, 5)	44	(10, 8)	62
(3, 3)	9	(6, 3)	21	(8, 2)	23	(9, 6)	48	(10, 9)	64
(4, 1)	8	(6, 4)	24	(8, 3)	29	(9, 7)	51	(10, 10)	65
(4, 2)	11	(6, 5)	26	(8, 4)	34	(9, 8)	53		
(4, 3)	13	(6, 6)	27	(8, 5)	38	(9, 9)	54		
(4, 4)	14	(7, 1)	14	(8, 6)	41	(10, 1)	20		
(5, 1)	10	(7, 2)	20	(8, 7)	43	(10, 2)	29		

It is evident that here, as in the case of the determination of plane curves by points, the resultant curve will be improper when any one of certain relations exists between the given points. Thus, if more than $p-q$ of the points chosen for the determination of a (p, q) lie on the double line in the first sheet of Σ, the curve must contain that line; and, similarly, if more than q of the $p(q+1)-\frac{1}{2}q(q-1)$ points lie on any generator of Σ, the curve determined must contain that generator; hence, if, in either of these cases, the curve be of order greater than unity, it will be an improper curve containing as a component a straight line. So, in general, when $p(q+1)-\frac{1}{2}q(q-1)$ points for the deter-

mination of a (p, q) are given, since a $(p-p', q-q')$ is determined by $(p-p')(q-q'+1)-\frac{1}{2}(q-q')(q-q'-1)$ points, it follows that when as many as

$$p(q+1)-\tfrac{1}{2}q(q-1)-(p-p')(q-q'+1)+\tfrac{1}{2}(q-q')(q-q'-1)$$
$$=p'(q-q'+1)+q'(p-q)+\tfrac{1}{2}q'(q'+1)$$

of the given points lie on a (p', q'), then the (p, q) is made up of the two curves (p', q') and $(p-p', q-q')$; but it is necessary here that $p' \leqq p$ and $q' \geqq q$, and, consequently, $p-p' \geqq q-q'$. Under these conditions it may be stated that, if, of the $p(q+1)-\frac{1}{2}q(q-1)$ points given for the determination of a (p, q), as many as $p'(q-q'+1)+q'(p-q)+\frac{1}{2}q'(q'+1)$ lie on a (p', q'), the (p, q) will consist of this (p', q'), together with the $(p-p', q-q')$ determined by the $(p-p')(q-q'+1)-\frac{1}{2}(q-q')(q-q'-1)$ remaining points.

Thus, if two of the $2p$ points given for the determination of a unicursal curve $(p, q)_{q=1}$ lie on a generator, the curve consists of that generator, together with the unicursal curve $(p-1, 1)$ determined by the $2(p-1)$ remaining points; and, in general, if $2p-\alpha$ of the $2p$ points given for the determination of a unicursal curve $(p, q)_{q=1}$ lie on a unicursal curve $(p-\alpha, 1)$, the α remaining points determine α generators, which, together with the $(p-\alpha, 1)$, constitute the $(p, q)_{q=1}$ in question. As a particular case, let there be given 50 points for the determination of a (10, 5); if 46 of those 50 points lie on a (6, 5), the 4 remaining points determine a (4, 0), 4 generators, which, with the (6, 5), constitute the (10, 5); or, if 42 of those 50 points lie on a (6, 4), the (10, 5) consists of that (6, 4), together with the (4, 1) determined by the 8 remaining points; or, again, if 39 of those 50 points lie on a (6, 3), the (10, 5) consists of that (6, 3), together with the (4, 2) determined by the 11 remaining points; or, similarly, if 37 of those 50 points lie on a (6, 2), the (10, 5) consists of that (6, 2), together with the (4, 3) determined by the 13 remaining points; if 36 of those 50 points lie on a (6, 1), the (10, 5) consists of that (6, 1), together with the (4, 4) determined by the 14 remaining points; if 48 of those 50 points lie on a (9, 4), the (10, 5) consists of that (9, 4), together with the (1, 1) determined by the 2 remaining points, etc., etc.

A case not in accord with what has been stated in this section occurs when the number of points prescribed by the formula for the determination of the curve is the same as the number of points in which two curves of the species of the

given curve intersect; i. e., when $p(q+1)-\frac{1}{2}q(q-1)=2pq-q^2$; this demands that $p=\frac{1}{2}\frac{q(q+1)}{q-1}$ and is satisfied by $p=3$ and $q=2$ or 3, but by no other values of p and q. Thus, a (3, 2) is determined by 8 points and two (3, 2)'s intersect in 8 points; if $\phi_1=0$ and $\phi_2=0$ are the equations of two (3, 2)'s which both pass through 7 given points, $\phi_1+k\phi_2=0$ is the equation of a (3, 2) passing through all the 8 intersections of the two given (3, 2)'s; consequently, all (3, 2)'s having 7 points common have in addition an 8th common point; and if the 8 points given for the determination of a (3, 2) are the points of intersection of two (3, 2)'s, the desired curve is not completely determined; for a single infinity of (3, 2)'s can be passed through the 8 given points, and one (3, 2) can be passed through the 8 given points and any 9th point. Similarly, and just as in the case of plane curves of the third order, a single infinity of (3, 3)'s can be passed through the 9 points of intersection of any two (3, 3)'s; and a 10th point is necessary to define a (3, 3) whenever the 9 given points are the points of intersection of two (3, 3)'s. Thus, if 32 of the 50 points given for the determination of a (10, 5) lie on a (7, 3), the (10, 5) consists of that (7, 3) and the (3, 2) determined by the 8 remaining points, if those 8 points are not the points of intersection of two (3, 2)'s; and, similarly, if 41 of the 50 points given for the determination of a (10, 5) lie on a (7, 2), the (10, 5) consists of that (7, 2) and the (3, 3) determined by the 9 remaining points, if those 9 points are not the points of intersection of two (3, 3)'s.

From the corresponding theorems in plane geometry are derived the following for the geometry on Σ:

a). If the two curves (p', q') and (p'', q'') have no multiple points of intersection,

1). Any $(p, q)_{q=p-\kappa}$ which passes through all the points of intersection of $(p', q')_{q'=p'}$ and $(p'', q'')_{q''\leqq p''}$, whose equations are $\phi'=0$ and $\phi''=0$, can have its equation put in the form $\phi\equiv A\phi'+B\phi''=0$, where $A=0$ and $B=0$ represent curves of the species $(p-p', q-q'-\varkappa)$ and $(p-p'', q-q''-\varkappa)$ respectively. Here must clearly $\varkappa\geqq q-q''-(p-p'')$.

2). Any $(p, q)_{q=p-(p'-q')-(p''-q'')+1}$, which passes through all the points of intersection of $(p', q')_{q'<p'}$ and $(p'', q'')_{q''<p''}$, whose equations are $\phi'=0$ and $\phi''=0$, can have its equation expressed in the form $\phi\equiv A\phi'+B\phi''=0$, where $A=0$ and $B=0$ represent curves of the species $(p-p', p-p'-(p''-q'')+1)$ and $(p-p'', p-p''-(p'-q')+1)$ respectively.

b). If the two curves (p', q') and (p'', q'') have an r-tuple point P of the former coincident with an s-tuple point P of the latter,

3). Any $(p, q)_{q=p-\kappa}$, which passes through all the points of intersection of $(p', q')_{q'=p'}$ and $(p'', q'')_{q''\leqq p''}$, whose equations are $\phi'=0$ and $\phi''=0$, will have an $(r+s-1)$-tuple point at P, and can have its equation expressed in the form $\phi \equiv A\phi' + B\phi'' = 0$, where $A=0$ represents a curve of the species $(p-p', q-q'-\kappa)$ having an $(s-1)$-tuple point at P, and $B=0$ represents a curve of the species $(p-p'', q-q''-\kappa)$ having an $(r-1)$-tuple point at P. Here evidently $\kappa \geqq q-q''-(p-p'')$.

4). Any $(p, q)_{q=p-(p'-q')-(p''-q'')+1}$, which passes through all the points of intersection of $(p', q')_{q'<p'}$ and $(p'', q'')_{q''<p''}$, whose equations are $\phi'=0$ and $\phi''=0$, will have an $(r+s-1)$-tuple point at P, and can have its equation expressed in the form $\phi \equiv A\phi' + B\phi'' = 0$, where $A=0$ represents a curve of the species $(p-p', p-p'-(p''-q'')+1)$ having an $(s-1)$-tuple point at P, and $A=0$ represents a curve of the species $(p-p'', p-p''-(p'-q')+1)$ having an $(r-1)$-tuple point at P.

Many interesting applications of these four theorems are possible. It follows from 1) that, if $pq''+p''q-qq''$ of the $pq'+p'q-qq'$ intersections of a $(p, q)_{q=p-\kappa}$ and a (p', q') lie on a $(p'', q'')_{q''\leqq p''}$, the remaining points of intersection lie on a $(p-p'', q-q''-\kappa)$; or, if pp'' of the $2pq-q^2$ intersections of two (p, q)'s lie on a $(p'', q'')_{q''=p''}$, the remaining points of intersection lie on a $(p-p'', q-q'')$. This means that, if 8 of the 16 points of intersection of two (4, 4)'s lie on a (2, 2), the remaining 8 points also lie on a (2, 2); if 6 of the 9 points of intersection of two (3, 3)'s lie on a (2, 2), the 3 remaining points lie on a (1, 1); and, again, if 17 of the 30 points of intersection of a (6, 5), with a (5, 5) lie on a (3, 2), the 13 remaining points lie also on a (3, 2).

From 2) it follows that, if $pq''+p''q-qq''$ of the $pq'+p'q-qq'$ points of intersection of a $(p, q)_{q<p}$ and a $(p', q')_{q'<p'}$ lie on a $(p'', q'')_{q''=p''-(p-q)+(p'-q')-1}$, the remaining points of intersection lie on a $(p-p'', p-p''-(p'-q')+1)$, where evidently $p'-q' \leqq (p-q)+1$; and if $pq''+p''q-qq''$ of the $2pq-q^2$ points of intersection of two $(p, q)_{q<p}$'s lie on a $(p'', q'')_{q''=p''-1}$, the remaining points of intersection lie on a $(p-p'', q-p''+1)$. Thus, if 5 of the 8 points of intersection of two (3, 2)'s lie on a (2, 1), the 3 remaining points lie on a (1, 1); and if 21 of the 30 points of intersection of a (6, 4) and a (6, 3) lie on a (4, 3), the 9 remaining points lie on a (2, 1).

Similar applications of Theorems 3) and 4) are readily obtained.

Theorem 1) shows also that any $(p, q)_{q=p}$, which passes through $p(q+1) - \frac{1}{2}q(q-1) - 1$ points of intersection of another $(p, q)_{q=p}$ and a $(p, q')_{q'=p-\kappa}$, contains also the $\frac{1}{2}(p-1)(p-2)$ remaining points of intersection if these do not all lie on a $(p-3, q-3)$. Thus any $(4, 4)$, which passes through 13 points of intersection of another $(4, 4)$ and a $(4, 2)$, contains also the 3 remaining points of intersection, unless those 3 points all lie on a $(1, 1)$; etc., etc.

4. *The Conic* $(1, 1)$ *on* Σ.

The most general linear equation in the three variables, λ, μ and ν, is of the form $a\lambda + b\mu + c\nu = 0$ and represents a curve of the species $(1, 1)$, which may be called a conic $(1, 1)$ on Σ. This curve meets every generator of Σ in one point, has consequently one point on the double line in the second sheet, but has no points on the double line in the first sheet of Σ. If $b = 0$, this conic passes through the point $x = 0$, $y = 0$, $z = 0$ in the second sheet of Σ, and, if $a = b = 0$, it is the conic which, with the generator $y = 0$, $z = 0$, forms the intersection of the plane $z = 0$ with Σ. If $a = 0$ and $b \neq 0$, this conic passes through the point $y = 0, z = 0, s = 0$. If $c = 0$, the curve is no longer a conic but a generator $(1, 0)$; but in this case it may be said that the conic consists of the generator in question and the double line in the first sheet of Σ, since the conic $(1, 1)$ tends to become that composite curve as a limiting case when c is made to approach zero; thus the $(1, 0)$ and the $(0, 1)$ together still constitute a conic $(1, 1)$ when $c = 0$, although the curve is no longer a proper conic.

A conic $(1, 1)$ is evidently determined by two points, neither of which lies on the double line in the first sheet of Σ; nor can both lie on any generator, if the curve is to be proper. The equation of the conic $(1, 1)$ through the points (1) and (2) may be written in the determinant form thus:

$$\begin{vmatrix} \lambda & \mu & \nu \\ \lambda_1 & \mu_1 & \nu_1 \\ \lambda_2 & \mu_2 & \nu_2 \end{vmatrix} = 0.$$

The condition that this conic pass through the pinch-point in the second sheet of Σ is obtained by putting $\lambda_1 = \mu_1 = 0$, which makes (1) become that point; this substitution gives:

$$\begin{vmatrix} \lambda & \mu & \nu \\ 0 & 0 & \nu_1 \\ \lambda_2 & \mu_2 & \nu_2 \end{vmatrix} = 0,$$

or $\mu_2\lambda - \lambda_2\mu = 0$, for the equation of the conic; but this equation gives only the generator $\lambda/\mu = \lambda_2/\mu_2$ through the given point (2); or, the improper conic in this case may be regarded as made up of this generator and the double line in the first sheet of Σ. This agrees with what was stated earlier, that no $(p, q)_{q=p}$ can pass through the pinch-point, and requires that point to be regarded as lying in the first sheet rather than in both sheets of Σ. If both points, (1) and (2), lie on the same generator, $\lambda_1/\mu_1 = \lambda_2/\mu_2$, the equation becomes $\mu_1\lambda - \lambda_1\mu = 0$; and, if that generator be the double line in the second sheet, $\lambda_1 = \lambda_2 = 0$, the equation becomes $\lambda = 0$; hence, whenever the two points lie on the same generator, the equation reduces to the equation of that generator, and the conic can be regarded as made up of that generator and the double line in the first sheet of Σ. Similarly, if one of the two points, as, e. g., the point (1), be chosen on the double line in the first sheet of Σ, the equation becomes that of the generator through the other point, viz., $\mu_2\lambda - \lambda_2\mu = 0$; and, if both points be chosen on the double line in the first sheet of Σ, the equation becomes $\lambda - \frac{\lambda_1 - \lambda_2}{\mu_1 - \mu_2} \cdot \mu = 0$, giving a generator determined by the two points, but containing neither of them so long as those points are distinct; hence the double line in the first sheet must always enter here to constitute, together with the generator found, the improper conic (1, 1) through, and determined by, the two points.

Evidently the coördinates of any point on the conic (1, 1) can be expressed linearly in terms of the coördinates of any two of its points thus:

$$\lambda = s\lambda_1 + t\lambda_2,$$
$$\mu = s\mu_1 + t\mu_2,$$
$$\nu = s\nu_1 + t\nu_2,$$

It has been seen already that two (1, 1)'s intersect in a single point. If the equations of the two conic (1, 1)'s on Σ be $a_1\lambda + b_1\mu + c_1\nu = 0$ and $a_2\lambda + b_2\mu + c_2\nu = 0$, the point of intersection of the two curves has the coördinates

$$\lambda : \mu : \nu = \begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix} : \begin{vmatrix} c_1 & a_1 \\ c_2 & a_2 \end{vmatrix} : \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}.$$

This point lies on the double line in the second sheet if $b_1/b_2 = c_1/c_2$, and cannot lie on that line in the first sheet of Σ so long as either conic (1, 1) is a proper curve.

In a similar way can all theorems concerning descriptive properties of lines

in the plane be applied to the case of the curves of species $(1, 1)$ on Σ. And in like manner can theorems concerning descriptive properties of curves of any order in the plane be applied to the cases of the corresponding curves on Σ.

5. *Polar and Tangent Curves on* Σ.

If the operator $\lambda\frac{\partial}{\partial\lambda'}+\mu\frac{\partial}{\partial\mu'}+\nu\frac{\partial}{\partial\nu'}$ be denoted by Δ, then may the curve whose equation is $\Delta^k\phi'=0$ be called the $(p-k)^{\text{th}}$ polar of the point (λ', μ', ν'), or P', with respect to the curve whose equation is $\phi=0$. If $k\leqq q$, this $(p-k)^{\text{th}}$ polar is a curve of the species (k, k), while, if $k\geqq q$, it is of the species (k, q); consequently, as a curve in space, the $(p-k)^{\text{th}}$ polar is a $2k$-thic or a $(k+q)$-thic, according as $k\leqq q$ or $k\geqq q$, and may be designated as the polar $2k$-thic or polar $(k+q)$-thic, respectively, of the point P' with respect to the curve (p, q). If the point P' be taken on the curve (p, q), the polar $2k$-thic and the polar $(k+q)$-thic become the tangent $2k$-thic and the tangent $(k+q)$-thic respectively; these curves have $k+1$ points lying on the curve (p, q) at the point P' and therefore give, in general, the direction of that curve at the point in question.

Thus, if P' be a point of the first order of multiplicity on (p, q) and do not lie on the double line in the second sheet of Σ, the direction of (p, q) is given most simply by the tangent conic, whose equation is $\Delta\phi'=0$. And, if P' be a point whose order of multiplicity is k on (p, q) and do not lie on the double line in the second sheet of Σ, the directions of the k branches are, in general, given by the tangent $2k$-thic or $(p+k)$-thic, whose equation is $\Delta^k\phi'=0$; but multiplicity must here be regarded as referring only to the intersections of branches all of which lie in the same sheet of Σ in the neighborhood in question.

If the equation of (p, q) is given in the form

$$\phi\equiv V_p+\lambda\,.\,V_{p-1}+\ldots\ldots+\lambda^{p-\iota-1}\,.\,V_{\iota+1}+\lambda^{p-\iota}\,.\,V_\iota=0,$$

the tangent conic at any point P' of the double line in the second sheet of Σ, that point being of the first order of multiplicity on (p, q), is a definite curve whose equation is

$\lambda\,.\,V'_{p-1}+\mu\,.\,\frac{\partial V'_p}{\partial\mu'}+\nu\,.\,\frac{\partial V'_p}{\partial\nu'}=0.$ If P' is not the pinch-point, $\frac{\partial V'_p}{\partial\nu'}\neq 0$, and the tangent conic cannot reduce to a tangent line. But if P' is the pinch-point, then must $p\geqq q+1$, and if V_p be written in the form:

$$V_p\equiv a_0\mu^p+a_1\mu^{p-1}\nu+\ldots\ldots+a_{q-1}\mu^{p-q+1}\nu^{q-1}+a_q\mu^{p-q}\nu^q,$$

it is necessary that $a_q = 0$ and $a_{q-1} \neq 0$ to insure that the curve pass once through the pinch-point in the second sheet. The pinch-point here is given by $\lambda' = 0$, $\mu' = 0$; V'_{p-1} contains the quantity μ' $p - q - 1$ times as a factor, while $\frac{\partial V'_p}{\partial \mu'}$ and $\frac{\partial V'_p}{\partial \nu'}$ contain the factors μ'^{p-q} and μ'^{p-q+1} respectively; hence the tangent curve at the pinch-point in the second sheet has for its equation $\lambda = 0$ and consists of the generator in that sheet, the double line itself; consequently any (p, q) passing once through the pinch-point in the second sheet must take the direction of the double line at that point. That such contact with the double line can occur, without having the curve meet the double line in the second sheet in two consecutive points, is evident, since the vanishing of a_q above shows that the terms containing ν^q all contain λ, and hence that the curve (p, q) has a point coincident with the pinch-point in the first sheet also; therefore a (p, q), passing once through the pinch-point in the second sheet, has contact with the double line at that point, the second point of intersection lying, not in the second, but in the first sheet there. Conversely, a (p, q) passing once through the pinch-point in the first sheet passes through the pinch-point in the second sheet also, and has the direction of the double line there, the two points of intersection with that line lying consecutively, one in one sheet and the other in the other sheet.

If the point P', determined on the double line in the second sheet of Σ by $\lambda' = 0$, $a\mu' - b\nu' = 0$, $(b \neq 0)$, is a double point of the curve (p, q), the equation of (p, q) can be written in the form

$$\phi \equiv (a\mu - b\nu)^2 . V_{p-2} + \lambda (a\mu - b\nu) . W_{p-2} + \lambda^2 . X_{p-2} + \dots + \lambda^{p-\iota} . X_\iota = 0 .$$

That P' be a double point demands that $q \geqq 2$; hence, the tangent curve here is a tangent quartic (2, 2) whose equation is

$$\lambda^2 X'_{p-2} + \lambda\mu . a W'_{p-2} + \mu^2 . a^2 V'_{p-2} - \lambda\nu . b W'_{p-2} - 2\mu\nu . ab V'_{p-2} + \nu^2 . b^2 V'_{p-2} = 0 .$$

If $X_{p-2} \equiv (a\mu - b\nu) . Y_{p-3}$, the equation of the tangent quartic becomes

$$[\lambda . W'_{p-2} + (a\mu - b\nu) . V'_{p-2}](a\mu - b\nu) = 0,$$

and one branch of the curve at P' has the direction of the conic (1, 1) whose equation is $a\mu - b\nu = 0$, while the other branch is tangent to the conic (1, 1) given by $\lambda . W'_{p-2} + \mu . a V'_{p-2} - \nu . b V'_{p-2} = 0$, the tangent quartic breaking up into two tangent conics of the species (1, 1). If $W_{p-2} \equiv (a\mu - b\nu) . Z_{p-3}$, the equation of the tangent quartic becomes $\lambda^2 . X'_{p-2} + (a\mu - b\nu)^2 . V'_{p-2} = 0$, and the

two branches of (p, q) have at P' the directions of the two conic (1, 1)'s given by

$$\lambda \cdot \sqrt{\frac{X'_{p-2}}{V'_{p-2}}} + a\mu - b\nu = 0 \text{ and } \lambda \cdot \sqrt{\frac{X'_{p-2}}{V'_{p-2}}} - a\mu + b\nu = 0.$$

And, if, again, it happen that not only $W_{p-2} = (a\mu - b\nu) \cdot Z_{p-3}$, but also $X_{p-2} \equiv (a\mu - b\nu) \cdot Y_{p-3}$, the equation of the tangent quartic reduces to $(a\mu - b\nu)^2 \cdot V'_{p-2} = 0$, and the tangent curve consists of the conic (1, 1) given by $a\mu - b\nu = 0$ occurring twice. So long as $b \neq 0$, it is clear that the curve cannot have contact with the generator $\mu = 0$, and hence cannot take the direction of the double line at any point of that line apart from the pinch-point. But if $b = 0$, and P' consequently be the pinch-point in the second sheet, the equation of the tangent quartic takes the form

$$\lambda^2 \cdot X'_{p-2} + \lambda\mu \cdot a W'_{p-2} + \mu^2 \cdot a^2 V'_{p-2} = 0,$$

in which the value zero is still to be introduced for μ'. It is clear that μ' occurs as a factor $p - q - 2$, $p - q - 1$ and $p - q$ times respectively in X'_{p-2}, W'_{p-2}, and V'_{p-2}; hence, the tangent curve consists of the double line itself in the second sheet as given twice by the equation $\lambda^2 = 0$. Here must V_p, as expressed on page 228, have $a_q = a_{q-1} = 0$, while $a_{q-2} \neq 0$; consequently, the terms of $\phi = 0$ containing ν^q all contain λ as a factor, and the curve (p, q), which passes twice through the pinch-point in the second sheet of Σ, contains the pinch-point in the first sheet at least once. Similar results are readily obtainable when P' is a point of any higher order of multiplicity on (p, q); and, in general, it is true that any branch of (p, q) which meets the double line in the second sheet at the pinch-point takes the direction of that line there, but at no other point of that line in the second sheet is the same true for any branch of (p, q).

If the equation of (p, q) be taken in the form

$$\phi \equiv U_p + \nu \cdot U_{p-1} + \cdots + \nu^{q-1} \cdot U_{p-q+1} + \nu^q \cdot U_{p-q} = 0,$$

the direction of the curve at any point P', of the first order of multiplicity on the curve and not lying on the double line in the first sheet of Σ, is given by the tangent conic (1, 1) at the point, the equation of that conic being

$$\lambda \cdot \frac{\partial \phi'}{\partial \lambda'} + \mu \cdot \frac{\partial \phi'}{\partial \mu'} + \nu \cdot \frac{\partial \phi'}{\partial \nu'} = 0.$$

This tangent conic becomes the generator at the point when $\frac{\partial \phi'}{\partial \nu'} = 0$ and $\frac{\partial \phi'}{\partial \lambda'} \neq 0$ or $\frac{\partial \phi'}{\partial \mu'} \neq 0$.

If P' lies on the double line in the first sheet: It has been seen on page 182 that the points where (p, q) meets that line are given by $U_{p-q} = 0$, and it is supposed that P' lies at the point where one of the $p - q$ generators given by that equation meets the double line, and that such an one of those points is chosen as shall be of the first order on (p, q) and shall not lie at the pinch-point. It is clear that the tangent curve to be found here is not a tangent conic $(1, 1)$ nor a tangent quartic $(2, 2)$, since those curves do not contain any one of the points in question; but the tangent must rather be a curve having $p \geqq q - 1$. To obtain the equation of this tangent curve at the point P', the following method is available: Since $\nu/\lambda = \nu/\mu = \infty$ all along the line in question, if $1/\lambda$, $1/\mu$ and $1/\nu$ be substituted for λ, μ and ν respectively in the equation $\phi = 0$, the problem becomes that of finding the tangent curve to the curve represented by the new equation thus obtained, at a point on the line given by $\nu/\lambda = \nu/\mu = 0$, which point, $\overline{P'}$ is of the, first order on the curve in question, and is itself given by $\nu = 0, \lambda/\mu = \rho$, where ρ is finite. This problem is analogous to that of finding the asymptotes of a plane curve. The new equation, obtained by the performance in $\phi = 0$ of the change of variables proposed above and cleared of fractions by multiplying by $\lambda^p \mu^p \nu^q$, may be designated by $\overline{\phi} = 0$ and has the form

$$\overline{\phi} \equiv \nu^q . \overline{U}_p + \nu^{q-1} . \overline{U}_{p+1} + \ldots\ldots + \nu . \overline{U}_{p+q-1} + \overline{U}_{p+q} = 0 .$$

The tangent curve at P' is evidently a tangent conic $(1, 1)$, in general, whose equation is

$$\lambda . \frac{\partial \overline{U}'_{p+q}}{\partial \lambda'} + \mu . \frac{\partial \overline{U}'_{p+q}}{\partial \mu'} + \nu . \overline{U}'_{p+q-1} = 0 .$$

U_{p-q} has the form

$$U_{p-q} \equiv (a_1 \lambda + b_1 \mu) . (a_2 \lambda + b_2 \mu) \ldots\ldots (a_{p-q} \lambda + b_{p-q} \mu) ,$$

and, accordingly,

$$\overline{U}_{p+q} \equiv \lambda^q \mu^q . (a_1 \mu + b_1 \lambda) . (a_2 \mu + b_2 \lambda) \ldots\ldots (a_{p-q} \mu + b_{p-q} \lambda) \equiv \lambda^q \mu^q . \overline{\overline{U}}_{p-q} ,$$

where $\overline{\overline{U}}_{p-q}$ denotes what U_{p-q} becomes if λ and μ are substituted for μ and λ respectively therein. Similarly, it is seen that

$$\overline{U}_{p+q-1} \equiv \lambda^{q-1} \mu^{q-1} . \overline{\overline{U}}_{p-q+1} ,$$

where $\overline{\overline{U}}_{p-q+1}$ has the same relation to U_{p-q+1} as $\overline{\overline{U}}_{p-q}$ to U_{p-q}. Hence, the

equation of the tangent conic above can be expressed in the form

$$\lambda \,.\, \frac{\partial\,(\lambda'^q \mu'^q \overline{\overline{U'_{p-q}}})}{\partial \lambda'} + \mu \,.\, \frac{\partial\,(\lambda'^q \mu'^q \overline{\overline{U'_{p-q}}})}{\partial \mu'} + \nu \,.\, \lambda'^{q-1} \mu'^{q-1} \overline{\overline{U'_{p-q+1}}} = 0,$$

or
$$\lambda \,.\, \frac{\partial \overline{\overline{U'_{p-q}}}}{\partial \lambda'} + \mu \,.\, \frac{\partial \overline{\overline{U'_{p-q}}}}{\partial \mu'} + \nu \,.\, \frac{\overline{\overline{U'_{p-q+1}}}}{\lambda' \mu'} = 0.$$

If, now, λ', μ', and ν' be substituted for $1/\lambda'$, $1/\mu'$, and $1/\nu'$ respectively here, and, likewise, λ, μ, and ν for $1/\lambda$, $1/\mu$, and $1/\nu$ respectively, the equation of the tangent conic (1, 1) to the curve given by $\overline{\phi} = 0$ at the point $\overline{P'}$ becomes the equation of the tangent cubic (2, 1) to the curve (p, q) at the point P', that equation being of the form

$$\lambda\mu \,.\, \frac{\overline{\overline{U'_{p-q+1}}}}{\lambda' \mu'} + \lambda\nu \,.\, \frac{\partial \overline{\overline{U'_{p-q}}}}{\partial \lambda'} + \mu\nu \,.\, \frac{\partial \overline{\overline{U'_{p-q+1}}}}{\partial \mu'} = 0.$$

If $\dfrac{U'_{p-q+1}}{\lambda'\mu'} = 0$ at P', the tangent cubic becomes a tangent line, the generator at P', given by $\lambda \,.\, \dfrac{\partial U'_{p-q}}{\partial \lambda'} + \mu \,.\, \dfrac{\partial U'_{p-q}}{\partial \mu'} = 0$; hence, if the group of terms U_{p-q+1} be lacking in the equation $\phi = 0$, the curve (p, q) has the direction of the generator at the point P'.

In a similar way, if P' lies at the pinch-point in the first sheet, it is found that the tangent cubic is reduced to the generator at the point given by $\lambda = 0$, i. e., the double line itself in the first sheet. As already seen, the curve (p, q) in this case passes through the pinch-point in the second sheet also, and can be regarded as having its two consecutive points on the double line lying one in either sheet at this point.

Similar results are obtained if the point P' is multiple on (p, q). And, in general, it may be said that the curve (p, q) has the direction of the double line at no point of that line save the pinch-point, but at the pinch-point can have no other direction.

6. *Plückerian Equations in the Geometry on Σ.*

The equation of the tangent conic $(1, 1)$ of any curve (p, q) at the point P' has been seen to be

$$\lambda \,.\, \frac{\partial \phi'}{\partial \lambda'} + \mu \,.\, \frac{\partial \phi'}{\partial \mu'} + \nu \,.\, \frac{\partial \phi'}{\partial \nu'} = 0.$$

As a locus in λ', μ', ν', this equation represents a curve of the species $(p - 1, q - 1)$

or $(p-1, q)$, according as $p=q$ or $p \geqq q+1$; let the equation of this curve, i. e., the above equation regarded as an equation in λ', μ', ν', be denoted by $\psi=0$. If the number of intersections of the two curves given by the equations $\phi=0$ and $\psi=0$ be denoted by N, then may N be defined as the class of the curve (p, q), since it gives the number of tangent conics which can be drawn from any point P on Σ to the curve (p, q), each point of intersection of the two curves being the point of contact of a tangent conic from the point P to the curve (p, q). If the point P lies on (p, q), it is clear that the number in question must be diminished by two for that point. And if Δ and K represent the number of double points and of cusps, respectively, occurring on (p, q), resulting from the intersections of branches lying in the same sheet in the neighborhood in question in each case, then, as in the corresponding case of plane curves, N must be subjected to a reduction by two for each double point and by three for each cusp, giving the formula

$$N=q(2p-q-1)-2\Delta-3K$$

for the class of any curve (p, q) on Σ. It is evident that Δ and K refer to the singularities designated on pages 214–217 by H and β.

A point of the curve (p, q) at which the tangent conic $(1, 1)$ meets that curve in three consecutive points may be called an inflexion on Σ. If the number of such points be denoted by I, a formula for I in terms of p and q can be found thus:

If H be defined by the determinant of the second derivatives of the polynomial ϕ:

$$H \equiv \begin{vmatrix} \frac{\partial^2\phi}{\partial\lambda^2} & \frac{\partial^2\phi}{\partial\lambda\partial\mu} & \frac{\partial^2\phi}{\partial\lambda\partial\nu} \\ \frac{\partial^2\phi}{\partial\mu\partial\lambda} & \frac{\partial^2\phi}{\partial\mu^2} & \frac{\partial^2\phi}{\partial\mu\partial\nu} \\ \frac{\partial^2\phi}{\partial\nu\partial\lambda} & \frac{\partial^2\phi}{\partial\nu\partial\mu} & \frac{\partial^2\phi}{\partial\nu^2} \end{vmatrix}$$

and the curve whose equation is $H=0$ be called the Hessian of (p, q), then will every intersection of this Hessian with the curve (p, q) be, in general, for that curve, an inflexion on Σ, as that term has been defined. The Hessian of (p, q) is of the species $(3p-6, 3q-2)$, if $p \geqq q+2$, $(3p-6, 3q-3)$, if $p=q+1$ and $(3p-6, 3q-6)$, if $p=q$, as the determinant above shows. Consequently the number of intersections of the curve (p, q) with its Hessian is found to be $2p(3q-1)-q(3q+4)$, when $p \geqq q+2$, and $3p(2q-1)-3q(q+1)$, when

$p \leqq q+1$. But these numbers are subject, as in the analogous case in plane curves, to a reduction by $6\Delta + 8K$, where Δ and K have the meanings assigned them above. And, furthermore, while the formula for the intersections used above makes the necessary reductions, in general, for the double line of Σ, it is known that a plane curve and its Hessian have contact between both branches of the two curves at a double point; accordingly, whenever $p \geqq q+2$, the curve (p, q) and its Hessian have $p-q$ actual intersections on the double line in the first sheet of Σ; but these intersections are no more to be regarded as inflexions on Σ for the curve (p, q) than are the points of intersection of a plane curve with the infinite line, in general, to be regarded as inflexions in determining the number ι for the curve in the plane; therefore, a further reduction by $p-q$ is necessary when $p \geqq q+2$. Thus is obtained the formula

$$I = 3p(2q-1) - 3q(q+1) - 6\Delta - 8K,$$

which holds for all values of p and q.

The same results are obtained at once from the Plückerian formulas for the class and the number of inflections of a plane curve if the curve is supposed to have a $(p-q)$-tuple point, and n, ι, δ, $\varkappa$, and m are replaced by N, I, Δ, K, and p respectively in the formulas for n and ι; thus

$n = m(m-1) - 2\delta - 3\varkappa$ gives

$N = p(p-1) - (p-q)(p-q-1) - 2\Delta - 3K$

$\quad = q(2p-q-1) - 2\Delta - 3K$; and, similarly,

$\iota = 3m(m-2) - 6\delta - 8\varkappa$ becomes

$I = 3p(p-2) - 3(p-q)(p-q-1) - 6\Delta - 8K$

$\quad = 3p(2q-1) - 3q(q+1) - 6\Delta - 8K.$

WILLIAMS COLLEGE, *February* 1, 1900.